Path Integral Approach (Based on Gauge Field Theory by S. Pokorski)

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Introduction in the framework of Quantum Mechanics

Schrödinger picture $\quad X_s|x\rangle = x|x\rangle$

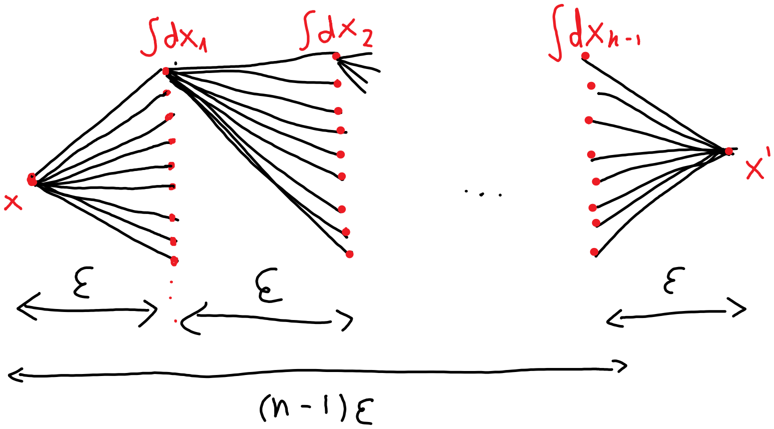
Heisenberg picture $\quad X_H(t)|x,t\rangle = x|x,t\rangle$

Related by $\quad |x,t\rangle = \exp\left(\frac{i}{\hbar}\hat{H}t\right)|x\rangle$

Matrix element corresponding to transformation from $|x\rangle$ at time t
To $|x'\rangle$ at time t' is given by:

$$\langle x',t'|x,t\rangle = \langle x'|\exp\left[\frac{i}{\hbar}\hat{H}(t-t')\right]|x\rangle \qquad (*)$$

We want to evaluate the matrix elements
For that, we split the time interval into n equal parts of duration ε



At each step, the propagation between neighboring points is given by

$$\langle x_j,t_j|x_{j-1},t_{j-1}\rangle = \langle x_j|\exp\left(-\frac{i}{\hbar}\varepsilon\hat{H}\right)|x_{j-1}\rangle$$

$$= \langle x_j|x_{j-1}\rangle - \frac{i\varepsilon}{\hbar}\langle x_j|\hat{H}|x_{j-1}\rangle + \mathcal{O}(\varepsilon^2)$$

By the virtue of the completeness relation $\quad \int dp\,|p\rangle\langle p| = 1$

We have

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$$\langle x_j|\hat{H}|x_{j-1}\rangle = \int dp_j \langle x_j|p_j\rangle\langle p_j|\hat{H}|x_{j-1}\rangle$$

$$= \int \frac{dp_j}{2\pi\hbar} \exp\left[\frac{i}{\hbar} p_j (x_j - x_{j-1})\right] \cdot H(p_j, x_{j-1})$$

We deal with classical Hamiltonian now! Additional integration over the momentum space- the particle may propagate between two points with arbitrary momenta.
(What about renormalization and Lorentz Invariace?)

We take the limit $n \to \infty$ and $\varepsilon \to 0$ also neglecting $\mathcal{O}(\varepsilon^2)$

Desired matrix element is given in terms of integral over all phase space.

$$\langle x',t'|x,t\rangle = \int \frac{\mathcal{D}x\mathcal{D}p}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}\int_t^{t'} [p\dot{x} - H(p,x)]\, d\tau\right\},$$

$$\int \mathcal{D}x\mathcal{D}p = \lim_{n\to\infty} \int \prod_{j=1}^{n} dx_j \prod_{j=1}^{n+1} dp_j$$

Now that's a lot of integrals.

Field Theory

1 degree of freedom

Continous *amount* of degrees of freedom labeled by $\vec{x}$

$$\begin{tikzcd} \text{clas. mech.} & & \text{QM} \\ x \in \mathbb{R} \arrow[rr] \arrow[d] & & \hat{X} \arrow[d] \\ \Phi(\vec{x},t) & & \hat{\Phi}(\vec{x},t) \\ \text{field theory} \arrow[urr] & & \text{QFT} \end{tikzcd}$$

QFT is usually formulated in terms of expectation values of time-ordered products of field operators

$$G^{(n)}(x_1, \ldots, x_n) \sim \langle 0|T\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)|0\rangle$$

vacuum

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Path Integral postulate:

Lorentz

$$G^{(n)}(x_1,\dots,x_n) \sim \int D\phi\ \phi(x_1)\dots\phi(x_n)\exp\left[\frac{i}{\hbar}\int d^4x_L \mathcal{L}\right]$$

Euclidean

$$G^{(n)}(x_1,\dots,x_n) \sim \int D\phi\ \phi(x_1)\dots\phi(x_n)\exp\left[-\frac{1}{\hbar}\int d^4x_E \mathcal{L}\right]$$

Related by Wick Rotation $\quad t \mapsto i\tau \quad$ analitical continuation

The normalization is

$$N = \langle 0|0\rangle \qquad G^{(n)}(x_1,\dots,x_n) = \frac{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle}{\langle 0|0\rangle}$$

$$N = \int D\phi\ \exp\left(\frac{i}{\hbar}\int d^4x\,\mathcal{L}\right)$$

One may calculate the path integral using the generating functional defined as

$$W[J] := \frac{1}{N}\int D\Phi\ \exp\left\{\frac{i}{\hbar}\int d^4x\left[\mathcal{L} + \hbar J(x)\phi(x)\right]\right\}$$

Then

$$G^{(n)}(x_1\dots x_n) = \left(\frac{1}{i}\right)^n \left.\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0} W[J]$$

functional derivative

Example- finding Green's function for the quadratic Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \frac{1}{2}i\varepsilon\phi^2$$

$\downarrow$ field equations

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$$\underbrace{\left(\partial_\mu \partial^\mu + m^2 - i\varepsilon\right)}_{\Box_x} \phi(x) = \hbar J(x)$$

The definition of Green's function gives shape to the solution

$$\Box_x G(x-y) := \delta(x-y)$$

$$\phi(x) = \hbar \int d^4y \, G(x-y) \, J(y)$$

This gives a differential equation for $G(x-y)$

$$\left(\partial_\mu \partial^\mu + m^2 - i\epsilon\right) G(x-y) = \delta(x-y)$$

$$\tilde{G}(k) = -\frac{1}{k^2 - m^2 + i\epsilon} \quad \longleftarrow \text{Propagator}$$

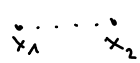
The generating functional is useful in the context of perturbation theory.
Consider an interaction Lagrangian, such that

$$S[\Phi] = S_0[\Phi] + S_I[\Phi]$$

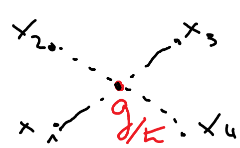
The *interacting* Green's function is given by

$$G^{(n)}(x_1 \ldots x_n) = \frac{\int \mathcal{D}\Phi \; \phi(x_1)\ldots\phi(x_n) \left[\sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{i}{\hbar} S_I\right)^N\right] \exp\left(\frac{i}{\hbar} S_0\right)}{\int \mathcal{D}\Phi \left[\sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{i}{\hbar} S_I\right)^N\right] \exp\left(\frac{i}{\hbar} S_0\right)}$$

$N = 0 \Rightarrow$ propagator

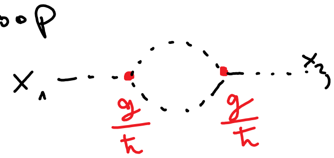


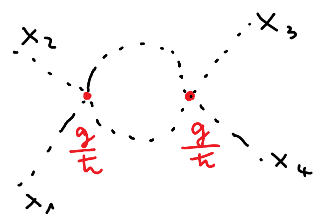
$N = 1 \Rightarrow$ tree level



$N = 2 \Rightarrow$ 1-loop

$\vdots$

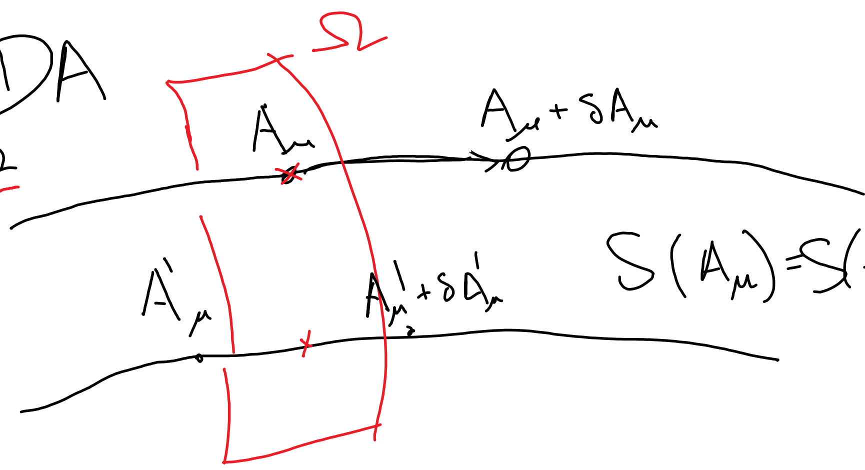




Fadeev - Popov ghosts

Faddeev – Popov ghosts
+
gauge fixing

$\int_{\Omega} DA$



$S(A_\mu) = S(A_\mu + \delta A_\mu)$